EXERCISES [MAI 4.13-4.14]

EXPECTATION ALGEBRA – CENTRAL LIMIT THEOREM

SOLUTIONS

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A. Paper 1 questions (SHORT)

EXPECTATION ALGEBRA

1.

(a)	(b)
$E(5X)=50$	$Var(5X)=100$
$E(X+Y+5)=30$	$Var(X+Y+5)=13$
$E(X-Y+5)=0$	$Var(X-Y+5)=13$
$E(3X+2Y)=60$	$Var(3X+2Y)=72$
$E(3X-2Y)=0$	$Var(3X-2Y)=72$

2. (a) (i) $E(X)=10$ and $Var(X)=10$ (ii) $E(Y)=8$ and $Var(Y)=8$

(b) (i) $E(X+Y)=18$ (ii) $Var(X+Y)=18$

(c) (i) $E(X-Y)=2$ (ii) $Var(X-Y)=18$

(d) $X+Y$ follows Poisson Po(18)

$X-Y$ does not follow any of the known distributions. It is just a discrete r.v.

(e) $P(X+Y=20)=0.0798$

3. (a) (i) $E(X)=10$ and $Var(X)=4$ (ii) $E(Y)=8$ and $Var(Y)=2$

(b) (i) $E(X+Y)=18$ (ii) $Var(X+Y)=6$

(c) (i) $E(X-Y)=2$ (ii) $Var(X-Y)=6$

(d) $X+Y$ follows $N(18,6)$ and $X-Y$ follows $N(2,6)$

(e) $P(X+Y\geq 20)=0.207$ (mind that $\mu=18$, $\sigma=\sqrt{6}$)

4. (a) (i) $E(X)=4$ and $Var(X)=2.4$ (ii) $E(Y)=3$ and $Var(Y)=2.4$

(b) (i) $E(X+Y)=7$ (ii) $Var(X+Y)=4.8$

(c) (i) $E(X-Y)=1$ (ii) $Var(X-Y)=4.8$

(d) $X+Y$ and $X-Y$ do not follow any of the known distributions. They are just discrete random variables.

(e) $P(X+Y=1)$ can be calculated only analytically (adding all possible cases:

$P(X+Y=1)=P(X=0)\times P(Y=1)+P(X=1)\times P(Y=0)$

$= 0.00604\times0.13194+0.04031\times0.03518 = 0.00222$

5. $\mathrm{E}(X)=10$ and $\mathrm{Var}(X)=10$

$\mathrm{E}(Y)=8$ and $\mathrm{Var}(Y)=8$

$\mathrm{E}(A)=40-40=0$

$\mathrm{Var}(A)=4^2\times 10+5^2\times 8=360$

6. $\mathrm{E}(X)=10$ and $\mathrm{Var}(X)=4$

$\mathrm{E}(Y)=8$ and $\mathrm{Var}(Y)=2$

$\mathrm{E}(A)=40-40=0$

$\mathrm{Var}(A)=4^2\times 4+5^2\times 2=114$

7. $\mathrm{E}(2Y+3)=6 \Leftrightarrow 2\mathrm{E}(Y)+3=6 \Leftrightarrow \mathrm{E}(Y)=\dfrac{3}{2}$

$\mathrm{Var}(2-3Y)=11 \Leftrightarrow 9\mathrm{Var}(Y)=11 \Leftrightarrow \mathrm{Var}(Y)=\dfrac{11}{9}$

8. $\mathrm{E}(R)=5$ and $\mathrm{Var}(R)=1$

$\mathrm{E}(S)=8$ and $\mathrm{Var}(S)=2$

$\mathrm{E}(V)=3\times 8-4\times 5=4$

$\mathrm{Var}(V)=3^2\times 2+4^2\times 1=34$

$\mathrm{P}(V>5)=0.432$ (mind that $\mu=4$, $\sigma=\sqrt{34}$)

9.

(i) $2\mu,\ 2\sigma^2$

(ii) $3\mu,\ 9\sigma^2$

(iii) $\mu,\ 3\sigma^2$

(iv) $\mu,\ \dfrac{\sigma^2}{n}$

10. Final answer: 0.944

11.

let X, Y, Z denote respectively the weights, in grams, of a randomly chosen apple, pear, peach

then $U=X+Y-2Z$ is $\mathrm{N}(115+110-2\times 105,\ 5^2+4^2+2^2\times 3^2)$

i.e. $\mathrm{N}(15,\ 77)$

we require

$\mathrm{P}(X+Y>2Z)=\mathrm{P}(U>0)$

$=0.956$

CENTRAL LIMIT THEOREM

12. (a) $E(\bar{X}) = 35$, $\text{Var}(\bar{X}) = \dfrac{25}{40} = 0.625$

(b) Since sample size > 30 we can assume by the CLT that

$$\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2) \text{ with } \mu_{\bar{X}} = 35 \text{ and } \sigma_{\bar{X}} = \frac{5}{\sqrt{40}} \cong 0.79057$$

(c) $P(34 \le \bar{X} \le 36) = 0.794$

13. (a) $E(X) = 35$, $\text{Var}(X) = 35$

$E(\bar{X}) = 35$, $\text{Var}(\bar{X}) = \dfrac{35}{40} = 0.875$

(b) Since sample size > 30 we can assume by the CLT that

$$\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2) \text{ with } \mu_{\bar{X}} = 35 \text{ and } \sigma_{\bar{X}} = \sqrt{0.875} \cong 0.93541$$

(c) $P(34 \le \bar{X} \le 36) = 0.715$

14. (a) $E(X) = 50 \times 0.7 = 35$, $\text{Var}(X) = 50 \times 0.7 \times 0.3 = 10.5$

$E(\bar{X}) = 35$, $\text{Var}(\bar{X}) = \dfrac{10.5}{40} = 0.2625$

(b) Since sample size > 30 we can assume by the CLT that

$$\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2) \text{ with } \mu_{\bar{X}} = 35 \text{ and } \sigma_{\bar{X}} = \sqrt{0.2625} \cong 0.51234$$

(c) $P(34 \le \bar{X} \le 36) = 0.949$

15. (a) $E(\bar{X}) = 35$, $\text{Var}(\bar{X}) = \dfrac{25}{40}$

(b) $\bar{X}$ is anyway Normal.

$$\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2) \text{ with } \mu_{\bar{X}} = 35 \text{ and } \sigma_{\bar{X}} = \frac{5}{\sqrt{40}} \cong 0.79057$$

(c) $P(34 \le \bar{X} \le 36) = 0.794$

16. (a) $\mu = 513.3$, $\sigma = 31.5$

Since sample size > 30 we can assume by the CLT that

$$\bar{X} \sim N(\mu_{\bar{X}}, \sigma_{\bar{X}}^2) \quad \mu_{\bar{X}} = 513.3 \qquad \sigma_{\bar{X}} = \frac{31.5}{\sqrt{40}} \cong 4.98059$$

(c) $P(510 \le \bar{X} \le 520) = 0.657$

B. Paper 2 questions (LONG)

17. (a) $A \sim N(30, 3^2)$, $P(A > 35) = 0.0478$

(b) let $X = B_1 + B_2 + B_3 + B_4 + B_5$

$E(X) = 5E(B) = 60$

$Var(X) = 5Var(B) = 20$

$P(X < 70) = 0.987$

(c) let $Y = A - 2B$

$E(Y) = E(A) - 2E(B) = 6$

$Var(Y) = Var(A) + 4Var(B) = 25$

$P(Y > 0) = 0.885$

18.

(a) let $W = \sum_{i=1}^{6} w_i$

w_i is $N(200, 15^2)$

$E(W) = \sum_{i=1}^{6} E(w_i) = 6 \times 200 = 1200$

$Var(W) = \sum_{i=1}^{6} Var(w_i) = 6 \times 15^2 = 1350$

W is $N(1200, 1350)$

$P(W > 1150) = 0.913$ by GDC

(b) let $W = 12w$

w is $N(80, 10^2)$

$E(W) = 12E(w) = 12 \times 80 = 960$

$Var(W) = 12^2 Var(w) = 12^2 \times 10^2 = 14400$

W is $N(960, 120^2)$

$P(W > 1150) = 0.0567$ by GDC

19. (a) probability = 0.691

(b) let X be the total weight of the 5 oranges

then $E(X) = 5 \times 205 = 1025$

$Var(X) = 5 \times 100 = 500$

$P(X < 1000) = 0.132$

(c) let $Y = B - 3C$ where B is the weight of a random orange and C the weight of a random lemon

$E(Y) = 205 - 3 \times 75 = -20$

$Var(Y) = 100 + 9 \times 9 = 181$

$P(Y > 0) = 0.0686$

20. Apples: $A \sim N(200, 15^2)$, Pears: $P \sim N(120, 10^2)$

(a) $A > 2P \Rightarrow A - 2P > 0$

Let $X = A - 2P$, $X \sim N(\mu, \sigma^2)$

$E(X) = 200 - 2 \times 120 = -40 = \mu$

$Var(X) = 15^2 + 4 \times 10^2 = 625 = \sigma^2$

$P(X > 0) = 0.0548$

(b) Let $Y = A_1 + A_2 + A_3 + P_1 + P_2 + P_3 + P_4$, $Y \sim N(\mu, \sigma^2)$

$E(Y) = 3 \times 200 + 4 \times 120 = 1080 = \mu$

$Var(Y) = 3 \times 15^2 + 4 \times 10^2 = 1075 = \sigma^2$

$P(Y > 1000) = 0.993$

21.

(a) let S be the weight of tea in a random *Supermug* tea bag

$S \sim \text{N}(4.2, 0.15^2)$

$\text{P}(S > 3.9) = 0.977$

(b) let M be the weight of tea in a random *Megamug* tea bag

$M \sim \text{N}(5.6, 0.17^2)$

$\text{P}(M > 5.4) = 0.880\ldots$

$\text{P}(M < 5.4) = 1 - 0.880\ldots = 0.119\ldots$

required probability $= 2 \times 0.880\ldots \times 0.119\ldots = 0.211$

(c) $\text{P}(S_1 + S_2 + S_3 + S_4 + S_5 < 20.5)$

let $S_1 + S_2 + S_3 + S_4 + S_5 = A$

$$\begin{aligned}\text{E}(A) &= 5\text{E}(S)\\ &= 21\end{aligned}$$

$$\begin{aligned}\text{Var}(A) &= 5\text{Var}(S)\\ &= 0.1125\end{aligned}$$

$A \sim \text{N}(21, 0.1125)$

$\text{P}(A < 20.5) = 0.0680$

(d) $\text{P}(S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 - (M_1 + M_2 + M_3 + M_4 + M_5) > 0)$

let $S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + S_7 - (M_1 + M_2 + M_3 + M_4 + M_5) = B$

$$\begin{aligned}\text{E}(B) &= 7\text{E}(S) - 5\text{E}(M)\\ &= 1.4\end{aligned}$$

$$\begin{aligned}\text{Var}(B) &= 7\text{Var}(S) + 5\text{Var}(M)\\ &= 0.302\end{aligned}$$

$\text{P}(B > 0) = 0.995$

22.

(a) (i) $\text{P}(X = 6) = 0.122$

(ii) $\text{P}(X = 6 \mid 5 \le X \le 8) = \dfrac{\text{P}(X = 6)}{\text{P}(5 \le X \le 8)} = \dfrac{0.122\ldots}{0.592\ldots - 0.0996\ldots}$

$= 0.248$

(b) (i) $\text{E}(\bar{X}) = 8$

$\text{Var}(\bar{X}) = \dfrac{8}{n}$

(ii) $\text{E}(\bar{X}) \neq \text{Var}(\bar{X})$ (for $n > 1$)

(c) $\bar{X} \sim \text{N}(8, 0.2)$

(i) $\text{P}(7.1 < \bar{X} < 8.5) = 0.846$

(ii) $\text{P}(\bar{X} \ge a) = 0.025 \Rightarrow a = 8.88$ (use InvN, tail: right)

(iii) $\text{P}(8 - k \le \bar{X} \le 8 + k) = 0.95 \Rightarrow 8 + k = 8.88$ (use InvN, tail: central)

$\Rightarrow k = 0.88$